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LETTER TO THE EDITOR

Self-similarity versus self-affinity: the Sierpinski gasket revisited

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Abstract. A Hilbert curve is constructed on the Sierpinski gasket. It is shown that despite the fact that the Sierpinski gasket is rigorously self-similar, the Hilbert curve is merely self-affine.

Until quite recently, it was commonly held that *self-similarity* is the essential attribute of a fractal structure, and which property posits the existence of a generally non-integral fractal dimension for that structure. Needless to say, this fractal dimension can be obtained from observing the similarity of the structure at any two well defined levels of resolution; consequently, it is really a *similarity* dimension. Thus, the similarity dimension of a Sierpinski gasket is log(3)/log(2), because at a given level of resolution the gasket contains three structures, each of which is exactly the gasket as observed when the resolution becomes coarser by a factor of two (Mandelbrot 1983). Most natural fractals, however, cannot show such a well ordered similarity; hence, various experimental and numerical procedures have been utilised in attempts to arrive at a fractal dimension for such structures like thin films (Messier and Yehoda 1985) and diffusion-limited aggregates (Witten and Sander 1981). It was soon found that unless the fractal was rigorously self-similar, fractal dimensions deduced from various procedures, e.g. from the mass-radius relationship (Meakin 1983) or from the density-density correlations (Meakin 1985), were not all equal, but hovering around, if indeed there were some elusive number whose existence could merely be conjectured.

There is now a growing debate on the attribute of self-similarity being the basis for the definition of fractal structures. Thus, on looking at the successive magnification stages of a thin-film surface (Messier and Yehoda 1985), it is readily apparent that they *look alike*, but they are certainly *not identical*. Hence, the notion of a similarity dimension is inapplicable for such a structure. Nonetheless, were one to concentrate on an appropriately small domain on the thin-film surface, one would obtain reasonably congruent values of *fractal dimensions* determined from different methodologies: one could then take the mean of the various fractal measures to arrive at a *local* fractal dimension. But this local dimension would vary from locale to locale on the same film, confounding thereby any attempt to construe a *global* fractal dimension, unless, of course, the film were to be merely a small perturbation of a rigorously self-similar structure. A similar conclusion has been mentioned by Lovejoy and Schertzer (1985) when they examined the stratification of clouds, and found that a large cloud could have several widely differing local fractal dimensions.

Consequent to the argument that such structures are neither rigorously self-similar nor are they perturbations of rigorously self-similar structures but they do possess small-range (fractal) order, it may be that self-similarity should not be considered the essential attribute of fractals. Instead, *self-affinity* should be, and self-similarity is a special case of this property. To understand self-affinity, consider a variant of the now famous question: how long is the coastline of continental Europe, excluding all islands? By measuring it as a function of the scale used to measure it, one can deduce some fraction which can be related to the *fractal dimension* of the European coastline. Let also the *fractal dimensions* of the coastlines of, say, France, Spain and Portugal be similarly ascertained. Inferences drawn from Richardson's experiments (Mandelbrot 1983) on several politico-geographical boundaries would have us believe that the fractal dimensions of these four coastlines could be all non-trivially different, but all lying within the same order of magnitude. And if the measurements are repeated for the coastlines of Artois, Normandy, Brittany and Gascony, still different *local fractal dimensions* would presumably be found. Yet most beaches look alike; one can conclude that the coastline of Europe is self-affine, but not self-similar. However, should one repeat this sequence on, say, a Koch triad (Mandelbrot 1983), one would end up with identical *local fractal dimensions*: this is due to the fact that the Koch triad is *self-similar*. Recently, Mandelbrot (1985, 1986) has elaborated on the difference between the attributes of self-similarity and self-affinity, and to which we refer the interested reader. Our aim here is simply to pose a question to the investigators in this area: can rigorously self-similar fractals also possess attributes that are merely self-affine?

In order to answer this question, we concentrate upon the well known Sierpinski gasket which is undoubtedly self-similar in the strictest sense. But, instead of modelling the gasket as a collage of triangles as is commonly done (Mandelbrot 1983), we represent it by a collection of nodes arranged on a triangular lattice. At a level of evolution L, $L \geqslant 1$, the Sierpinki gasket consists of 2^L rows, the inter-row distance being a, while the nodes on any given row are separated from each by distances which are multiples of $2b$, $0 < \tan^{-1}(b/a) < \pi/2$. It is convenient to describe the Sierpinski gasket, modelled as a collection of nodes, by the recursive function f_L defined by

$$f_L(x, y) = f_{L-1}(x, y) * g_L(x, y) \tag{1}$$

where * is a convolution operator (Goodman 1968), the initiator is

$$f_1(x, y) = \delta\{x, y\} + \delta\{x - a, y - b\} + \delta\{x - a, y + b\} \tag{2}$$

the generator is

$$g_L(x, y) = \delta\{x, y\} + \delta\{x - a2^{L-1}, y - b2^{L-1}\} + \delta\{x - a2^{L-1}, y + b2^{L-1}\} \tag{3}$$

and $\delta\{\ \}$ is the Dirac delta function. It is easy to see that if the gasket evolves from stage L to stage $L+1$, then its maximum linear dimension, measured in terms of the occupied rows, doubles; but the number of nodes increases by a factor of 3, regardless of L; hence, it possesses a similarity dimension of $\log(3)/\log(2)$.

Proceeding with these collections of nodes, we join them together with straight-line segments in order to connect all nodes on the gasket of a given level L. Care must be taken to ensure that those nodes of the triangular grid which are not part of the gasket do not lie on the curve thus formed. In addition, only two line segments should meet at any given node on the gasket, with the exception of the two extremal nodes. In this way, the curve formed does not have tendrils hanging out from it and neither does it contain any closed loops. It will be found that only two curves satisfying these conditions are possible, one of which is illustrated in figure 1, and the other is its mirror reflection. Both curves are Hilbert curves, being space-filling in nature; their slopes are discontinuous at (3^L-2) points.

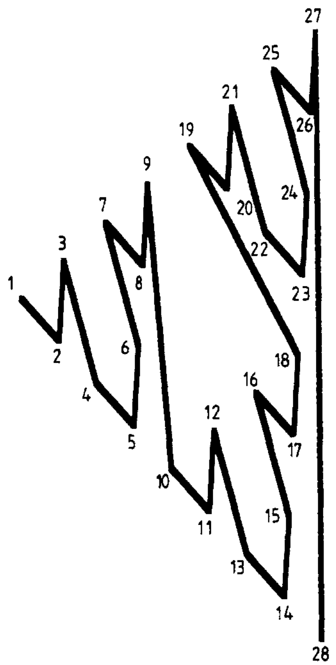

Figure 1. Illustration of the construction of a Hilbert curve on a Sierpinski gasket.

Are these curves themselves fractals? It certainly would appear so because the rules for generating are not only rigorously precise, but they are also recursive over the gasket evolutionary levels. In figures 2 and 3, the Hilbert curves are shown for levels $L=4$ and 6, respectively, when $a=b=1$. It may be observed from there that they are certainly related very strongly; not surprisingly, because they form the *medulla oblongata* of rigorously self-similar gaskets. In order to obtain a fractal dimension, if any, of this curve, we computed the length Λ_L of the Hilbert curve for different values of L and defined a descriptor

$$R_{L/L-1}=\log\{\Lambda_L/\Lambda_{L-1}\}/\log(2). \tag{4}$$

Shown in table 1 are the computed values of $R_{L/L-1}$ when $a=b=1$. It is observed from this table that as L increases, then $R_{L/L-1}$ asymptotically tends to reach a value of log(3)/log(2), the similarity dimension of the Sierpinski gasket. It can be easily shown that

$$\Lambda_L=3\Lambda_{L-1}+\surd[a^2+(2^L-1)^2b^2]+\surd[(2^{L-1}-1)^2a^2+(2^{L-1}+1)^2b^2] \tag{5a}$$

$$\Lambda_1=2b+\surd[a^2+b^2] \tag{5b}$$

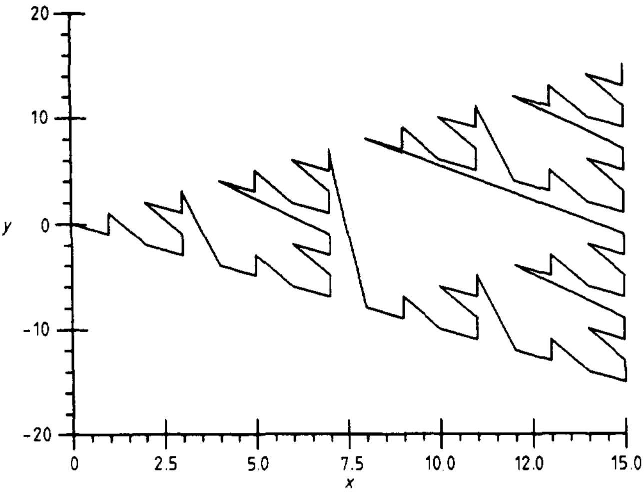

Figure 2. Hilbert curve drawn on a Sierpinski gasket of level $L=4$; $a=b=1$.

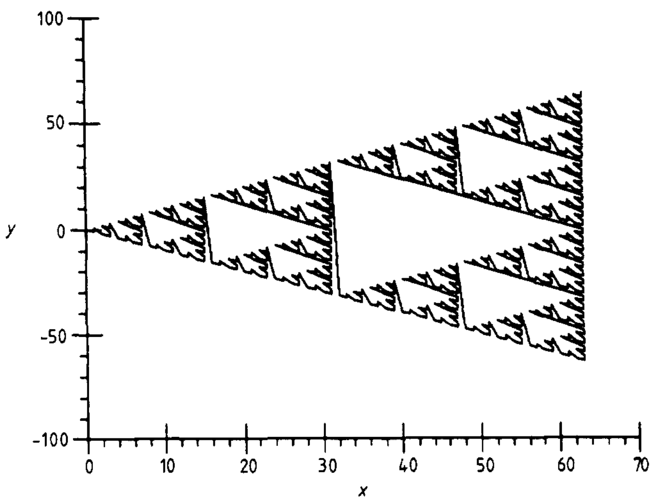

Figure 3. Hilbert curve drawn on a Sierpinski gasket of level $L=6$; $a=b=1$.

from which it is easy to deduce the limit of $R_{L/L-1}$ as L goes to infinity. The reason why it is never equal to the similarity dimension of the gasket is because of the presence of the line segments joining the lower level gaskets which together form the gasket of the higher level, e.g. the line segments joining nodes 3 and 4, and joining nodes 9 and 10 in figure 1. As L increases the contribution of such line segments to Λ_L decreases, and Λ_L/Λ_{L-1} tends to 3.

The self-affinity of the Hilbert curve, thus, may not be doubted. Note, however, should be taken of the fact that a Hilbert curve of level L contains the Hilbert curves of all of the lower levels. Depending upon the ability to resolve the curve, it will be found that the local descriptors are quite different from the global descriptor.

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Table 1. Fractal descriptors $R_{L/L-1}$ computed for a Hilbert curve on a Sierpinski gasket for which $a = b$. Note that the similarity dimension of the Sierpinski gasket is $\log(3)/\log(2) = 1.584\,96$

L	$R_{L/L-1}$
2	2.278 70
3	1.917 92
4	1.774 95
5	1.700 70
6	1.657 86
7	1.631 80
8	1.615 43
9	1.604 94
10	1.598 14
15	1.586 66
20	1.585 19
25	1.584 99
30	1.584 97
32	1.584 96
35	1.584 96
45	1.584 96
60	1.584 96

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